Exercise 8

In Exercises 5–8, show that the given function $u(x)$ is a solution of the corresponding Volterra integral equation:

$$u(x) = 1 + 2x + \sin x + x^2 - \cos x - \int_0^x u(t)\,dt,\ u(x) = 2x + \sin x$$

Solution

Substitute the function in question on both sides of the integral equation.

$$2x + \sin x \stackrel{?}{=} 1 + 2x + \sin x + x^2 - \cos x - \int_0^x (2t + \sin t)\,dt$$

Subtract $2x + \sin x$ from both sides.

$$\begin{aligned}
0 &\stackrel{?}{=} 1 + x^2 - \cos x - \left(\int_0^x 2t\,dt + \int_0^x \sin t\,dt\right) \\
&\stackrel{?}{=} 1 + x^2 - \cos x - \left[t^2\Big|_0^x + (-\cos t)\Big|_0^x\right] \\
&\stackrel{?}{=} 1 + x^2 - \cos x - [x^2 + (-\cos x) - (-\cos 0)] \\
&\stackrel{?}{=} 1 + x^2 - \cos x - x^2 + \cos x - 1 \\
&= 0
\end{aligned}$$

Therefore,

$$u(x) = 2x + \sin x$$

is a solution of the Volterra integral equation.